PMM U.S.S.R.,Vol.50,No.2,pp.165-170,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00
© 1987 Pergamon Journals Ltd.

NEW THEOREMS ON FORCED PERIODIC OSCILLATIONS IN NON-LINEAR CONTROL SYSTEMS*

A.M. KRASNOSEL'SKII

A general method is given for obtaining criteria, expressed in one-sided estimates of the non-linearities, for the existence and uniqueness of periodic oscillations, as well as criteria for the realizability and convergence of the harmonic balance procedure for an approximate construction of the periodic oscillations. The method is based on special (and not, apparently, previously mentioned) spectral properties of the operator of the periodic problem for a linear section with a bilinear transfer function.

1. The properties of the symmetric part of the periodic-problem operator. We consider a linear section W, see e.g., /1–4/ with transfer function $W(p) = M(p) / L(p)$, where

$$M(p) = b_0p^m + b_1p^{m-1} + \ldots + b_m,\quad L(p) = p^l + a_1p^{l-1} + \ldots + a_l \tag{1.1}$$

and polynomials (1.1) are relatively prime, have real coefficients, and $m < l$. We will denote by W_α the section with transfer function

$$W_\alpha(p) = M(p) / [L(p) - \alpha M(p)] \tag{1.2}$$

Assume that $\omega_k i$ $(k = 0, 1, \ldots)$, where $\omega_k = 2k\pi T^{-1}$ and T is a fixed positive number, are not roots of the equation

$$L(p) - \alpha M(p) = 0 \tag{1.3}$$

Then, corresponding to any T-periodic input $u(t)$ of W_α we have a unique T-periodic output $x(t) = A_\alpha u(t)$, where

$$A_\alpha u(t) = \int_0^T G_\alpha(t - s; T)\, u(s)\, ds \tag{1.4}$$

and $G_\alpha(t;\ T)$ is the corresponding pulse-frequency response /3–5/ of section W_α. We know that the operator (1.4) acts from space $L_1 = L_1(0, T)$ into space $C = C(0, T)$ and is continuous; it acts in space $L_2 = L_2(0, T)$, is normal in the usual sense /6/, and is completely continuous; it is also completely continuous as an operator from L_2 into space $C^{l-m-1} = C^{l-m-1}(0, T)$ of functions which are continuous along with their derivatives up to order $l - m - 1$ and satisfy the condition of T-periodicity. Simple calculation shows that, with any $k = 0, 1, \ldots,$ we have the equations

$$\begin{aligned} A_\alpha(\xi \cos \omega_k t + \eta \sin \omega_k t) = {} & [\xi \operatorname{Re} W_\alpha(\omega_k i) + \\ & \eta \operatorname{Im} W_\alpha(\omega_k i)] \cos \omega_k t + [\eta \operatorname{Re} W_\alpha(\omega_k i) - \xi \operatorname{Im} W_\alpha(\omega_k i)] \times \\ & \sin \omega_k t \end{aligned} \tag{1.5}$$

Assume that E_0 is a one-dimensional subspace of function-constants, E_k $(k = 1, 2, \ldots)$ is a two-dimensional subspace with basis of functions $\cos \omega_k t, \sin \omega_k t$, P_k is the orthogonal projector in L_2 onto E_k. We shall here consider the symmetric part $B_\alpha = {}^1/_2 (A_\alpha + A_\alpha^*)$ of opertor (1.4). It follows from (1.5) that

$$B_\alpha u = \sum_{k=0}^{\infty} \operatorname{Re} W_\alpha(\omega_k i)\, P_k u \quad (u \in L_2) \tag{1.6}$$

Operator (1.6) is selfadjoint and completely continuous (since $m < l$) in L_2. If it is non-negative definite, its fractional powers are defined /6/:

$$B_\alpha^\gamma u = \sum_{M(\omega_k i) \neq 0} [\operatorname{Re} W_\alpha(\omega_k i)]^\gamma P_k u \tag{1.7}$$

With $\gamma > 0$ these fractional powers are completely continuous in L_2, while if $\gamma < 0$ they are unbounded operators; they commute in a natural way with operator (1.4). We put

$$H_\alpha = B_\alpha^{-1/2} A_\alpha = A_\alpha B_\alpha^{-1/2} \tag{1.8}$$

An important role is played below by the polynomial

$$\Pi(\omega) = \operatorname{Re}[M(\omega i) L(-\omega i)] \ (-\infty < \omega < \infty) \tag{1.9}$$

of even degree $2\pi(M, L)$, and by the quantities

*Prikl.Matem.Mekhan.,50,2,224-230,1986

$$\alpha_-(T) = \inf_{k=0,1,\ldots;M(\omega_k i)\neq 0} \Pi(\omega_k) / |M(\omega_k i)|^2 \tag{1.10}$$

$$\alpha_+(T) = \sup_{k=0,1,\ldots;M(\omega_k i)\neq 0} \Pi(\omega_k) / |M(\omega_k i)|^2 \tag{1.11}$$

If $\pi(M, L) \leqslant m$ (e.g., when $m = l-1$), both quantities (1.10) and (1.11) are finite; if $\pi(M, L) > m$ (e.g., when $l-m$ is even), only one, (1.10) or (1.11), is finite.

Theorem 1. Let the quantity (1.10) be finite and $\alpha < \alpha_-(T)$. Then we have the following four assertions:

1°. Numbers $\omega_k i$ $(k = 0, 1, 2, \ldots)$ are not roots of Eq. (1.3), and hence operators (1.4) and (1.6) are defined.

2°. Operator (1.6) is non-negative definite.

3°. Operator (1.8) acts and is continuous in space L_2.

4°. If the extra condition $\pi(M, L) > m$ holds, then (1.8) acts from space L_2 into space $C^{\pi(M,L)-m-1}$ and is completely continuous.

Proof. If some $\omega_k i$ were a solution of Eq. (1.3), it would follow, since $M(p)$ and $L(p)$ are relatively prime, that $M(\omega_k i) \neq 0$. Hence we should have

$$\alpha = L(\omega_k i) / M(\omega_k i) = \Pi(\omega_k) / |M(\omega_k i)|^2 \geqslant \alpha_-(T)$$

which contradicts the hypotheses. This proves the first claim.

Consider the eigenvalues

$$\mu_k(\alpha) = \mathrm{Re} W_\alpha(\omega_k i) = [\Pi(\omega_k) - \alpha\, |M(\omega_k i)|]\, \Delta_k^{-2} \tag{1.12}$$
$$(\Delta_k = |L(\omega_k i) - \alpha M(\omega_k i)|)$$

of operator (1.6). If $M(\omega_k i) = 0$, then $\mu_k(\alpha) = 0$. If $M(\omega_k i) \neq 0$, then

$$\mu_k(\alpha) = [\Pi(\omega_k) / |M(\omega_k i)|^2 - \alpha]\, |M(\omega_k i)|^2 \Delta_k^{-2} \geqslant [\alpha_-(T) - \alpha]\, |M(\omega_k i)|^2 \Delta_k^{-2} > 0$$

so that the second claim holds.

Consider the polynomial $\Pi(\omega) - \alpha\, |M(\omega i)|^2$. Its degree is not less than $2m$, since otherwise the equation

$$\lim_{\omega\to\infty} \frac{\Pi(\omega) - \alpha\,|M(\omega i)|^2}{|M(\omega i)|^2} = 0$$

would imply a contradiction $\alpha \geqslant \alpha_-(T)$ with the hypotheses. Hence for eigenvalues (1.12) we have

$$\overline{\lim_{k\to\infty}} \frac{k^{-2(l-m)}}{\mu_k(\alpha)} < \infty$$

whence it follows that

$$\overline{\lim_{k\to\infty}} \frac{|W_\alpha(\omega_k i)|}{\sqrt{\mu_k(\alpha)}} < \infty \tag{1.13}$$

Using Eqs. (1.5), we can write operator (1.4) as

$$A_\alpha u = \sum_{k=0}^{\infty} |W_\alpha(\omega_k i)|\, U_k(\alpha) P_k u \qquad (u \in L_2) \tag{1.14}$$

where each operator $U_k(\alpha)$ acts and is isometric in the corresponding subspace E_k. By (1.14), operator (1.8) is

$$H_\alpha u = \sum_{k=0}^{\infty} \nu_k(\alpha) U_k(\alpha) P_k u \qquad (u \in L_2) \tag{1.15}$$

where $\nu_k(\alpha) = 0$, if $M(\omega_k i) = 0$, and $\nu_k(\alpha) = |W_\alpha(\omega_k i)|\, [\mu_k(\alpha)]^{-1/2}$, if $M(\omega_k i) \neq 0$. By (1.13), we have $|\nu_k(\alpha)| \leqslant c < \infty$ $(k = 0, 1, \ldots)$. Hence

$$\|H_\alpha u\|^2 = \sum_{k=0}^{\infty} \nu_k^2(\alpha)\, \|U_k(\alpha) P_k u\|^2 = \sum_{k=0}^{\infty} \nu_k^2(\alpha)\, \|P_k u\|^2 \leqslant c^2 \|u\|^2 \qquad (u \in L_2)$$

i.e., the thrid assertion holds.

Let the extra condition $\pi(M, L) > m$ hold. Then the degree of polynomial $\Pi(\omega) - \alpha\, |M(\omega i)|^2$ is the same as $2\pi(M, L)$ and we have the relation, stronger than (1.13):

$$\overline{\lim_{k\to\infty}} |W_\alpha(\omega_k i)|\, k^{\pi(M,L)-m} / \sqrt{\mu_k(\alpha)} < \infty \tag{1.16}$$

We put

$$Ru = \sum_{k=0}^{\infty} (1+k)^{-\pi(M,\,L)+m} P_k u \qquad (u \in L_2) \tag{1.17}$$

Operator (1.17) is completely continuous (as is easily proved by the methods given in /7/), as an operator from L_2 into space $C^{\pi(M,\,L)-m-1}$. From (1.15) we have the equation $H_\alpha = RH_\alpha{}^0$, in which

$$H_\alpha{}^0 u = \sum_{k=0}^{\infty} (1+k)^{\pi(M,\,L)-m} \nu_k(\alpha) U_k(\alpha) P_k u \qquad (u \in L_2) \tag{1.18}$$

Now, to prove the fourth claim, it only remains to note that, by (1.16), operator (1.18) is continuous in L_2.

When proving the fourth claim we have used the continuity of H_α^0 as an operator from L_2 into $C^{\pi(M,\,L)-m-1}$. Operator (1.18) in fact acts from L_2 into narrower functional spaces, so that the fourth claim could obviously be strengthened; but we shall not do this, since it involves introducing special terminology.

The theorem is easily modified for the case when the quantity (1.11) is finite and we have $\alpha > \alpha_+(T)$.

2. Forced oscillations in non-linear single-circuit systems. Consider a system consisting of a linear section W, embraced by non-linear feedback. The problem on the forced oscillations of this system is described, see e.g., /1/, by the equation

$$L\left(\frac{d}{dt}\right) x = M\left(\frac{d}{dt}\right) f(t, x) \tag{2.1}$$

The function $f(t, x)$ is assumed to be T-periodic in t and continuous in the set of its variables. The assumption of continuity is introduced to simplify the treatment; without this assumption, we have to use concepts such as superpositional measurability, Caratheodory conditions, and occasional passage to generalized solutions, etc.

We introduce the notation

$$w(T, \alpha) = \sup_{k=0,\,1,\,2,\ldots} \left| \frac{M(\omega_k i)}{L(\omega_k i) - \alpha M(\omega_k i)} \right| \tag{2.2}$$

If the quantity (2.2) is finite, and

$$| f(t, x) - \alpha x | \leqslant c\, | x | + c_1 \tag{2.3}$$

where $cw(T, \alpha) < 1$, then Eq.(2.1) has at least one T-periodic solution (a "T-solution"), which is continuous along with its derivatives up to order $l - m$. This assertion follows at once from the results of /5/. The main role is played below, not by conditions of type (2.3), but by one-sided constraints on the non-linearity.

We shall say that the non-linearity $f(t, x)$ is correctly matched with section W if either number (1.10) is finite and

$$xf(t, x) \leqslant \alpha x^2 + \alpha_1 \ (-\infty < t,\ x < \infty),\ \alpha < \alpha_-(T) \tag{2.4}$$

or else number (1.11) is finite and

$$xf(t, x) \geqslant \alpha x^2 - \alpha_1 \ (-\infty < t,\ x < \infty),\ \alpha > \alpha_+(T) \tag{2.5}$$

Theorem 2. Let non-linarity $f(t, x)$ be correctly matched with the linear section W. If $\pi(M, L) \leqslant m$, where $2\pi(M, L)$ is the degree of polynomial (1.9), we assume that, in addition,

$$| f(t, x) | \leqslant hx^2 + h_1 \ (-\infty < t,\ x < \infty) \tag{2.6}$$

Then, Eq.(2.1) has at least one T-solution, which is continuous along with its derivatives up to order $l - m$.

Proof. We shall confine ourselves to the case when the number (1.10) is finite and inequalities (2.4) hold. Since $\alpha < \alpha_-(T)$, then, by the first assertion of Theorem 1, operator (1.4) is defined. Hence the problem on the T-solutions of Eq.(2.1) is equivalent to the equation (A_α is the linear integral operator (1.4))

$$x(t) = A_\alpha F_\alpha x(t),\ \ F_\alpha x(t) = f[t,\ x(t)] - \alpha x(t) \tag{2.7}$$

The solution of Eq.(2.7) will be sought in the form $x(t) = A_\alpha y(t)$. Then, the function $x(t)$ will be an $l - m$ times continuously differentiable T-solution of Eq.(2.1), if $y(t)$ is a continuous solution of the equation

$$y(t) = F_\alpha A_\alpha y(t) \tag{2.8}$$

Since the operator $F_\alpha A_\alpha$ acts and is completely continuous in space C, the Leray-Schauder principle /8/ can be used to prove that Eq.(2.8) is solvable in C. For this, we prove a common a priori estimate for the norms in C of all solutions of all equations, with $\lambda \in [0, 1]$

$$y(t) = \lambda F_\alpha A_\alpha y(t) \tag{2.9}$$

Let $y_*(t) \in C$ be the solution of (2.9) with $\lambda = \lambda_* \in [0, 1]$. We multiply equation $y_*(t) = \lambda_* F_\alpha A_\alpha y_*(t)$ scalarly (in the sense of the scalar product in L_2) by the function $A_\alpha y_*(t)$. By (2.4), we have the estimate

$$(y_*(t), A_\alpha y_*(t)) = \lambda_* \int_0^T \{f[s, A_\alpha y_*(s)] - \alpha A_\alpha y_*(s)\} A_\alpha y_*(s)\, ds \leqslant T\alpha_1$$

But, by the second assertion of Theorem 1,

$$(y_*(t), A_\alpha y_*(t)) = (y_*(t), B_\alpha y_*(t)) = \| B_\alpha^{1/2} y_*(t) \|_{L_2}^2$$

Hence

$$\| B_\alpha^{1/2} y_*(t) \|_{L_2} \leqslant \sqrt{T\alpha_1} \tag{2.10}$$

We first assume that $\pi(M, L) > m$. Then, by (2.30) and the fourth assertion of Theorem 1, we have

$$\| A_\alpha y_*(t) \|_C \leqslant \| H_\alpha \|_{L_2 \to C} \| B_\alpha^{1/2} y_*(t) \|_{L_2} \leqslant \| H_\alpha \|_{L_2 \to C} \sqrt{T\alpha_1} = \alpha_2 < \infty$$

whence we have the a priori estimate

$$| y_*(t) | \leqslant \max_{\tau \in [0, T],\, |u| \leqslant \alpha_2} | f(\tau, u) - \alpha u | = \alpha_3 < \infty.$$

Now let $\pi(M, L) \leqslant m$. From the third assertion of Theorem 1 and (2.10), we have

$$\| A_\alpha y_*(t) \|_{L_2} \leqslant \| H_\alpha \|_{L_2 \to L_2} \| B_\alpha^{1/2} y_*(t) \|_{L_2} \leqslant \| H_\alpha \|_{L_2 \to L_2} \sqrt{T\alpha_1} = \alpha_4$$

whence, by (2.6),

$$\| F_\alpha A_\alpha y_*(t) \|_{L_1} \leqslant h \| A_\alpha y_*(t) \|_{L_2}^2 + h_1 T + |\alpha| \int_0^T | A_\alpha y_*(s) | ds \leqslant$$
$$h\alpha_4^2 + h_1 T + |\alpha| \alpha_4 \sqrt{T} = \alpha_5$$

i.e., $\| y_*(t) \|_{L_1} \leqslant \alpha_5$. But the operator A_α acts from L_1 into C and is continuous; hence

$$\| A_\alpha y_*(t) \|_C \leqslant \| A_\alpha \|_{L_1 \to C}\, \alpha_5 = \alpha_6$$

and we have the a priori estimate

$$\| y_*(t) \|_C \leqslant \max_{\tau \in [0, T],\, |u| \leqslant \alpha_6} | f(\tau, u) - \alpha u | = \alpha_7 < \infty.$$

If the function $f(t, x)$ is not continuous, to prove the solvability of Eq.(2.8) we apply the Leray-Schauder principle according to the same scheme. But Eq.(2.8) then has to be considered in spaces other than C (e.g., in space L_2).

3. Criteria for the uniqueness of forced oscillations.

If the number (2.2) is finite and

$$| f(t, x) - f(t, y) - \alpha(x - y) | \leqslant c\, | x - y | \tag{3.1}$$
$$(-\infty < t,\ x,\ y < \infty)$$

where $cw(T, \alpha) < 1$, then the uniqueness of the T-solution of Eq.(2.1) is easily obtained from the principle of contraction mappings. Theorem 1 leads to uniqueness criteria of a different type.

We shall say that $f(t, x)$ satisfies a one-sided Lipschitz condition, matched with section W, if either number (1.10) is finite and

$$(x - y)[f(t, x) - f(t, y)] \leqslant \alpha(x - y)^2 \quad (-\infty < t,\ x, y < \infty) \tag{3.2}$$
$$\alpha < \alpha_-(T)$$

or else number (1.11) is finite and

$$(x - y)[f(t, x) - f(t, y)] \geqslant \alpha(x - y)^2 \quad (-\infty < t,\ x, y < \infty) \tag{3.3}$$
$$\alpha > \alpha_+(T)$$

Conditions (3.2) and (3.3) are equivalent to $f(t, x) - \alpha x$ being suitable monotonic with respect to x. If $f(t, x)$ is differentiable with respect to x, condition (3.2) is equivalent to the bound $f_x'(t, x) \leqslant \alpha$, and (3.3) to the bound $f_x'(t, x) \geqslant \alpha$.

Theorem 3. Let $f(t, x)$ satisfy a one-sided Lipschitz condition matched with linear section W. Then, Eq.(2.1) has not more than one continuous T-solution.

Proof. We shall confine ourselves to the case when the number (1.10) is finite and condition (3.2) holds. Let P_* be the orthogonal projector from the linear hull E_* of all E_k with numbers k for which $M(\omega_k i) = 0$. Obviously, $\dim E_* \leqslant m$.

Let $x_1(t)$ and $x_2(t)$ be T-solutions of Eq.(2.1). From the equations $x_1(t) = A_\alpha F_\alpha x_1(t)$,

$x_2(t) = A_\alpha F_\alpha x_2(t)$ it follows that, for $y_1(t) = F_\alpha x_1(t)$, $y_2(t) = F_\alpha x_2(t)$, we have the relations $y_1(t) = F_\alpha A_\alpha y_1(t)$, $y_2(t) = F_\alpha A_\alpha y_2(t)$.

By Theorem 1, we obtain from (3.2) the estimate

$$\| B_\alpha^{1/2}(y_1 - y_2)\|_{L_2}^2 = (y_1 - y_2, B_\alpha(y_1 - y_2)) = (y_1 - y_2, A_\alpha(y_1 - y_2)) =$$
$$\int_0^T [F_\alpha A_\alpha y_1(s) - F_\alpha A_\alpha y_2(s)][A_\alpha y_1(s) - A_\alpha y_2(s)]\, ds \leqslant 0$$

Hence $B_\alpha^{1/2} y_1(t) = B_\alpha^{1/2} y_2(t)$. It therefore follows that $(I - P_*)\, y_1(t) = (I - P_*) y_2(t)$, i.e., $(I - P_*)\, F_\alpha x_1(t) = (I - P_*) \cdot F_\alpha x_2(t)$. But $A_\alpha P_* = 0$; hence $A_\alpha F_\alpha x_1(t) = A_\alpha F_\alpha x_2(t)$, i.e., the functions $x_1(t)$ and $x_2(t)$ are identical.

4. Application of the harmonic balance method. The harmonic method (HBM) is widely used in control theory, see e.g., /3, 9-11/ and the quoted references. It originates from the work of Bogolyubov and Krylov, and amounts to a projection procedure.

Denote by R_N the orthogonal projector $R_N = P_0 + P_1 + \ldots + P_N$ onto the $(2N + 1)$-dimensional subspace $E(N)$ of trigonometric polynomials which contain only harmonics $\sin \omega_k t$, $\cos \omega_k t$ $(k = 0, 1, \ldots, N)$. In HBM we fix an integer $N > 0$ and seek the approximate T-solution of Eq.(2.1) as a trigonometric polynomial $x_N(t) \in E(N)$; its unknown $2N + 1$ coefficients are chosen in such a way that $x_N(t)$ is a solution of the equation

$$L\left(\frac{d}{dt}\right) x = M\left(\frac{d}{dt}\right) R_N f(t, x) \tag{4.1}$$

We can regard (4.1) as a system of $2N + 1$ scalar equations. An HBM is called realizable if (4.1) has a solution for any N. As usual in the theory of projection procedures for solving non-linear equations, it is said to be convergent with respect to the norm of space E if, as $N \to \infty$, the Hausdorff distance in E from the set of solutions of Eq.(4.1) to the set of all T-solutions of Eq.(2.1) tends to zero.

If the operator (1.4) is defined, the problem of the T-solutions of Eq.(4.1) is equivalent to the operator equation $x(t) = A_\alpha R_N F_\alpha x(t)$, similar to Eq.(2.7); and this equation can be studied by the constructions of Sects.2 and 3 (see also /5/). We shall confine ourselves to making one assertion.

Theorem 4. In the conditions of Theorem 2, the HBM for obtaining the forced oscillations is realizable and converges with respect to the norm of space C^{l-m}. In the conditions of Theorem 3, Eq.(4.1) for any N has a unique T-solution.

When proving the realizability of HBM in the conditions of Theorem 2, no use is made of estimate (2.6).

The above constructions, connected with one-sided estimates of non-linearities (like the constructions of /5/, connected with two-sided estimates), can be extended to more complicated systems than (2.1), e.g., to systems with delay (regarding such systems see e.g., /12/), to systems with derivative-based controls, and to those with hysteresis, etc.

REFERENCES

1. VORONOV A.A., Foundations of automatic control theory (Osnovy teorii avtomaticheskogo upravleniya), Energiya, Moscow, 1980.
2. KALMAN R.E. et al., Topics in mathematical system theory, McGraw, 1968.
3. ROZENVASSER E.N., Oscillations of non-linear systems. Method of integral equations (Kolebaniya nelineinykh sistem. Metod integral'nykh uravnenii), Nauka, Moscow, 1969.
4. KRASNOSEL'SKII M.A., POKROVSKII A.V. and PRIVAL'SKII V.B., Conditions for pulse-frequency responses to be positive, Avtomat. i Telemekhan., 9, 1980.
5. KRASNOSEL'SKII A.M., Frequency criteria in the problem of the forced oscillations of automatic control systems, Avtomat. i telemekhan., 9, 1980.
6. DUNFORD N. and SCHWARTZ J.T., Linear operators, 2, Spectral theory. Selfadjoint operators in Hilbert space, Wiley, 1968.
7. KRASNOSEL'SKII M.A., ZABREIKO P.P., PUSTYL'NIK E.I. and SOBOLEVSKII P.E., Integral operators in spaces of summable functions (integral'nye operatory v prostranstvakh summiruemykh funktsii), Nauka, Moscow, 1966.
8. LERAY J. and SCHAUDER U., Topology and functional equations, Usp. mat. Nauk., 3-4, 1946.
9. POPOV E.P. and PAL'TOV I.P., Approximate methods for studying non-linear automatic systems (Priblizhennye metody issledovaniya nelineinykh avtomaticheskikh sistem), Fizmatgiz, Moscow, 1960.
10. BRAVERMAN E.M., MEERKOV S.M. and PYATNITSKII E.S., The small parameter in the problem of justifying the harmonic-balance method (in case of filter hypothesis), Avtomat. i telemekhan., 1, 5-21; 2, 5-12, 1975.

11. KRASNOSEL'SKII M.A. et al., Approximate solution of operator equations (Priblizhennoye reshenie operatornykh uravnenii), Nauka, Moscow, 1969.
12. KOLMANOVSKII V.B. and NOSOV V.R., Stability and periodic modes of controlled systems with after-effect (Ustoichivost' i periodichenskie rezhimy reguliruemykh sistem s posledeistviem), Nauka, Moscow, 1981.

Translated by D.E.B.

PMM U.S.S.R.,Vol.50,No.2,pp.170-174,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00
© 1987 Pergamon Journals Ltd.

ON OPTIMAL PROBLEMS OF THE THEORY OF ELASTICITY WITH UNKNOWN BOUNDARIES*

L.V. PETUKHOV

In optimal-design problems in the theory of elasiticty when the shape of the boundary is sought /1—3/, the domain varies and in the long run is subject to determination, unlike design problems when the elastic moduli of the material are unknown /4, 5/. The solution of such problems is often irregular in nature /2/. In this connection, the need arises for a classification of a suitable set of allowable domains that can be given by using two parameters. For this set of domains a variational concept is presented and a theorem is proved on the existence of variations of the displacements of an elastic structure.

1. The class of domains under consideration. Let R^n be an n-dimensional Euclidean space of vectors $\mathbf{x}$ ($n = 2$ or 3) in which a Cartesian coordinate system is defined by the directions $\mathbf{e}_i$ such that $\mathbf{x} = x_i\mathbf{e}_i$. Here and henceforth, the Latin subscripts take the values $1, \ldots, n$; summation from 1 to n is assumed over the repeated subscripts in the products.

Definition 1. The set /6/

$$\Gamma = \{\mathbf{y} \in R^n \mid \mathbf{y} = \mathbf{y}(\tau^\circ),\ \tau^\circ \in T\}$$
$$T = \{\tau^\circ = (\tau^2, \ldots, \tau^n) \mid 0 < \tau^\alpha < 1, 0 < \sum_{\alpha=2}^{n} \tau^\alpha < 1\}$$
$$\mathbf{y}_k(\tau^\circ) \in C^m(\bar{T}),\ \ m \geqslant 1$$

is called a differentiable $(n-1)$-dimensional cell, when C^m is a space of m times differentiable functions, the vectors $\mathbf{r}_\alpha = \partial\mathbf{y} \setminus \partial\tau^\alpha$ are linearly independent for $\forall\tau^\circ \in \bar{T}$, i.e., form a covariant moving basis of the coordinate system τ^α /7/, the mapping $\mathbf{y}_k(\tau^\circ)$ is one-to-one in $\bar{T}$. Here and henceforth, the Greek indices take the values $2, \ldots, n$, and summation from 2 to n is assumed in the repeated super- and subscripts in the products.

We determine the normal direction $\mathbf{r}_1 = \mathbf{r}_1(\mathbf{y})$ at each point $\mathbf{y} \in \Gamma$, where we select its direction such that

$$Y = Y(\tau^\circ) = \det \| \mathbf{r}_1\mathbf{r}_2 \ldots \mathbf{r}_n \| > 0 \tag{1.1}$$

The area element of the surface Γ is determined by the formula /8/

$$d\Gamma = Y(\tau^\circ)\, d\tau^2 \ldots d\tau^n = Y(\tau^\circ)\, d\tau^\circ$$

We introduce curvilinear coordinates /7/

$$\mathbf{x}(\tau) = \mathbf{y}(\tau^\circ) + \tau^1\mathbf{r}_1(\mathbf{y}(\tau^\circ)),\ \tau = (\tau^1, \ldots, \tau^n) \tag{1.2}$$

in the neighbourhood of Γ and we calculate the covariant vectors

$$\mathbf{R}_\alpha(\tau) = \partial\mathbf{x}/\partial\tau^\alpha = \mathbf{r}_\alpha - \tau^1\mathbf{r}_\alpha \cdot \mathbf{t},\quad \mathbf{R}_1(\mathbf{y}) = \mathbf{r}_1(\mathbf{y}) \tag{1.3}$$

where $\mathbf{t} = \mathbf{t}(\mathbf{y}(\tau^\circ)) = -\nabla^\circ\mathbf{r}_1$ is the curvature tensor of the surface Γ, $\nabla^\circ = \mathbf{r}^\alpha\, \partial/\partial\tau^\alpha$ is an $(n-1)$-dimensional Hamilton operator, and $\mathbf{r}^\alpha$ is the contravariant basis of the coordinate system τ^α. The tensor $\mathbf{t}$ and the direction $\mathbf{r}_1$ depend only on the cells Γ but not on the selection of the coordinates τ°.

Let $X(\tau) = \det \| \mathbf{R}_1\mathbf{R}_2 \ldots \mathbf{R}_n \|$ be the Jacobi matrix of the coordinate transformation $\mathbf{x}(\tau)$. It follows from (1.3) that

$$X(\tau) = Y(\tau^\circ)\, [1 + I_1(\mathbf{t})\,(-\tau^1) + \ldots + I_{n-1}(\mathbf{t})\,(-\tau^1)^{n-1}]$$

**Prikl.Matem.Mekhan.*,50,2,231-236,1986